Models for the Hop-constrained Steiner Tree Problem

AUTHORS: Bernard Fortz (ULB), Luís Gouveia and Pedro Moura (CMAFcIO, ULisboa)

R&D UNIT: CMAFcIO

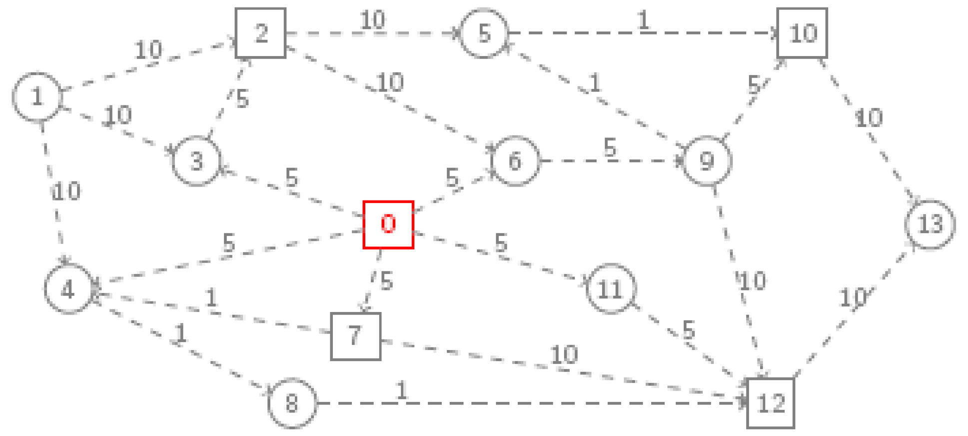
CONTACT: pmmoura@fc.ul.pt

The Problem (HSTP)

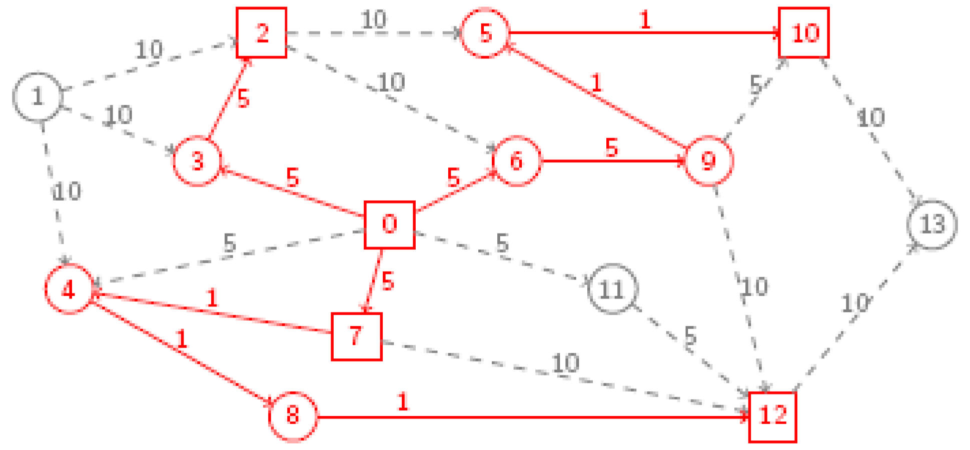
- Given
  - a directed graph $G = (N, A)$ with:
    - node set, $N = \{0,1,..,n\}$,
    - arc set, $A = \{(i,j): i,j \in N\}$ and a function $c_{ij} > 0, \forall (i,j) \in A$
    - required nodes set, $R \subseteq N, 0 \in R$
  - Hop limit, $H \in \mathbb{N}$
- Find an arborescence $G' = (S, A(S))$, rooted at 0 with $S \subseteq N, A(S) \subseteq A$:
  - $R \subseteq S, |path_{0 \to r}| \leq H$ arcs , $\forall r \in R$
  - $C(A(S)) = \sum_{(i,j) \in A(S)} c_{ij}$ is minimum

Examples

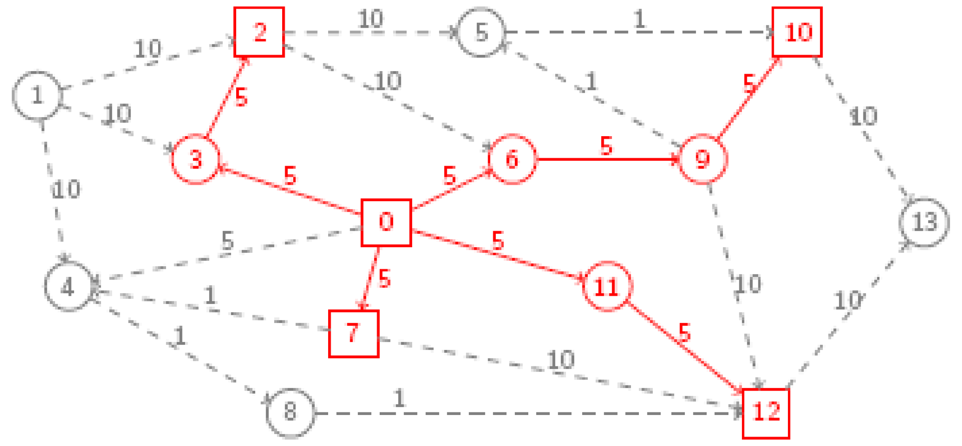
- $H = 2$: no solution



- $H = 3$: $C(A(S)) = 40$



- $H = 4$: $C(A(S)) = 35$



The Node Hop-indexed Model

$$i \in N \setminus \{0\}: \quad y_i = 1, \text{ if } i \in S \text{ (0, otherwise)}$$
$$(i,j) \in A: \quad x_{ij} = 1, \text{ if } (i,j) \in A(S) \text{ (0, otherwise)}$$
$$i \in N \setminus \{0\}, h \in \{1\} \cup \mathbb{H}: \quad v_i^h = 1, \text{ if } i \text{ is at distance } h \text{ from the root (0, otherwise)}$$

$$\sum_{(i,j) \in A} x_{ij} = y_j \quad j \in N \setminus \{0\} \tag{1}$$
$$\sum_{h \in \{1\} \cup \mathbb{H}} v_i^h = y_i \quad i \in N \setminus \{0\} \tag{2}$$
$$v_j^1 = x_{0j} \quad (0,j) \in A \tag{3}$$
$$v_i^{h-1} + x_{ij} \leq v_j^h + 1 \quad (i,j) \in A,\ i \in N \setminus \{0\},\ h \in \mathbb{H} \tag{4}$$
$$v_i^H + x_{ij} \leq 1 \quad (i,j) \in A,\ i \in N \setminus \{0\} \tag{5}$$
$$y_i = 1 \quad i \in R \setminus \{0\} \tag{6}$$
$$x_{ij} \in \{0,1\} \quad (i,j) \in A \tag{7}$$
$$y_i \in \{0,1\} \quad i \in N \setminus \{0\} \tag{8}$$
$$v_i^h \in \{0,1\} \quad i \in N \setminus \{0\},\ h \in \{1\} \cup \mathbb{H} \tag{9}$$

Result

Model (1) - (9) is not a valid formulation for the HSTP since conectivity between the root node and the required nodes is not guaranteed

Strenghening and generalizing constraints (4):

$$v_i^H + \sum_{h \in M} v_i^{h-1} + x_{ij} \leq \sum_{h \in M} v_j^h + y_i \quad i \in N \setminus \{0\},\ (i,j) \in A,\ M \subseteq \mathbb{H} \tag{4+}$$

New Valid Inequalities (symmetric version of (4)):

$$v_j^h + x_{ij} \leq v_i^{h-1} + 1 \quad i \in N \setminus \{0\},\ (i,j) \in A,\ h \in \mathbb{H} \tag{10}$$

Result

Model (NHB): (1) - (3), (5) - (9), (10) is a valid formulation for the HSTP

Strenghening and generalizing constraints (10):

$$v_j^1 + \sum_{h \in Q} v_j^h + x_{ij} \leq y_j + \sum_{h \in Q} v_i^{h-1} \quad i \in N \setminus \{0\},\ (i,j) \in A,\ Q \subseteq \mathbb{H} \tag{10+}$$

Model (NHB^+): (1) - (3), (5) – (9), **(10+)**

The Arc Hop-indexed Model

$$i \in N \setminus \{0\}: \quad y_i = 1, \text{ if } i \in S \text{ (0, otherwise)}$$
$$(i,j) \in A: \quad x_{ij} = 1, \text{ if } (i,j) \in A(S) \text{ (0, otherwise)}$$
$$(i,j) \in A, h \in \{1\} \cup \mathbb{H}: \quad z_{ij}^h = 1, \text{ if } (i,j) \text{ is in position } h \text{ in the } path_{0 \to j} \text{ (0, otherwise)}$$

(AH) model

$$\sum_{(i,j) \in A} x_{ij} = y_j \quad j \in N \setminus \{0\} \tag{1}$$
$$z_{0j}^1 = x_{0j} \quad j: (0,j) \in A \tag{11}$$
$$\sum_{h \in \mathbb{H}} z_{ij}^h = x_{ij} \quad i \in N \setminus \{0\},\ (i,j) \in A \tag{12}$$
$$\sum_{(k,i) \in A, k \neq 0} z_{ki}^{h-1} \geq z_{ij}^h \quad i \in N \setminus \{0\},\ (i,j) \in A,\ h \in \mathbb{H},\ h \geq 3 \tag{13}$$
$$z_{0i}^1 \geq z_{ij}^2 \quad i,j \in N \setminus \{0\}: (0,i),(i,j) \in A \tag{14}$$
$$y_i = 1 \quad i \in R \setminus \{0\} \tag{6}$$
$$x_{ij} \in \{0,1\} \quad (i,j) \in A \tag{7}$$
$$y_i \in \{0,1\} \quad i \in N \setminus \{0\} \tag{8}$$
$$z_{ij}^h \in \{0,1\} \quad i \in N \setminus \{0\},\ (i,j) \in A,\ h \in \mathbb{H} \tag{15}$$
$$z_{0j}^1 \in \{0,1\} \quad (0,j) \in A \tag{16}$$

Result

Model (AH) dominates (NHB^+) in terms of the Linear Program Relaxation

Strenghening constraints (13) by removing z_{ji}^{h-1} from the summation:

$$\sum_{(k,i) \in A, k \neq 0, j} z_{ki}^{h-1} \geq z_{ij}^h \quad i \in N \setminus \{0\},\ (i,j) \in A,\ h \in \mathbb{H},\ h \geq 3 \tag{13+}$$

Model (AH^+): (1), (6) - (8), (11), (12), **(13+)**, (14), (15), (16)

Results

- **Instances**: Symmetric complete graphs with 20, 40 and 80 nodes, 20%, 40% and 60% required nodes and H =2,3,4,5
- **Tested models**: (NHB^+), (AH) and (AH^+)
- **Linear Programming gaps**:
  - increase with increasing H, decreasing number of required nodes
  - the weaker arc model (AH) reduces the node model (NHB^+) LP gaps in 0% - 17%
  - the stronger arc model (AH^+) reduces the arc model (AH) LP gaps in 0% - 12%
- **CPU times**:
  - LP relaxation: 0 – 40 sec.; Integral models: arc models less than 30 sec., (NHB^+) spent more time (some ≅ 1 hour)

References

- L. Gouveia, *Using hop-indexed models for constrained spanning and Steiner tree models*, 1999, Telecommunications network planning, Springer
- L. Gouveia, M. Leitner and M. ruthmair, *Layered graph approaches for combinatorial optimization problems*, 2019, Comp. & OR
- M. Sinnl and I. Ljubic, *A node-based layered graph approach for the STP with revenues, budget and hop-constraints*, 2016, Math. Program. Comp.